Math 155 Review from Calculus I **Name**____________

Revisit Integration and Differentiation

Section 4.5 Integration by Substitution

THEOREM 4.12 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Do you see a derivative?

Ex.1 Integrate: $\int t^3\sqrt{t^4+5}\,dt$

Let $u = t^4+5$

$\frac{du}{dt} = 4t^3$

$du = \frac{du}{dt} \cdot dt$

$du = 4t^3\,dt$

$\frac{du}{4t^3} = dt$

$= \int t^3 \cdot u^{1/2} \cdot \left(\frac{du}{4t^3}\right)$

$= \frac{1}{4} \cdot \int u^{1/2}\,du$

$= \frac{1}{4} \cdot \left[\frac{2}{3} \cdot u^{3/2}\right] + C$

$= \frac{1}{6} u^{3/2} + C$

$\boxed{= \frac{1}{6}(t^4+5)^{3/2} + C}$

check: $\frac{d}{dt}\left[\frac{1}{6}(t^4+5)^{3/2} + C\right] = \frac{1}{6} \cdot \left[\frac{3}{2} \cdot (t^4+5)^{1/2}\right] \cdot \frac{d}{dt}(t^4+5) + 0$

$= \frac{1}{4} \cdot (t^4+5)^{1/2} \cdot (4t^3)$

$= t^3(t^4+5)^{1/2}$ ✓

Do you see a derivative?

Ex.2 Solve: $\dfrac{dy}{dx} = \dfrac{10x^2}{\sqrt{1+x^3}}$

$$\int\left(\frac{dy}{dx}\right)dx = \int\left(\frac{10x^2}{\sqrt{1+x^3}}\right)dx$$

$$\int dy = \int 10x^2(1+x^3)^{-1/2}\,dx$$

Let $u = 1+x^3$

$\dfrac{du}{dx} = 3x^2$

$du = \dfrac{du}{dx}\cdot dx$

$du = 3x^2dx$

$\dfrac{du}{3x^2} = dx$

$$y = 10\cdot\int x^2\cdot(u)^{-1/2}\cdot\left(\frac{du}{3x^2}\right)$$

$$y = \frac{10}{3}\cdot\int u^{-1/2}du$$

$$y = \frac{10}{3}\cdot\left[\frac{2}{1}\cdot u^{1/2}\right] + C$$

$$y = \frac{20}{3}u^{1/2} + C$$

$$\boxed{y = \frac{20}{3}(1+x^3)^{1/2} + C}$$

check! $\dfrac{d}{dx}\left[\dfrac{20}{3}(1+x^3)^{1/2} + C\right] = \dfrac{d}{dx}[y]$

$$\frac{dy}{dx} = \frac{20}{3}\cdot\frac{1}{2}(1+x^3)^{-1/2}\cdot\frac{d}{dx}[1+x^3] + 0$$

$$\frac{dy}{dx} = \frac{10}{3}\cdot(1+x^3)^{-1/2}\cdot(3x^2)$$

$$\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}} \quad ✓$$

Do you see a derivative?

Ex.3 Integrate: $\int \sec^2(x)\sqrt{\tan(x)}\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$du = \frac{du}{dx} \cdot dx$

$du = \sec^2(x)\,dx$

$\frac{du}{\sec^2(x)} = dx$

$= \int \sec^2(x) \cdot (u)^{1/2} \cdot \left(\frac{du}{\sec^2(x)}\right)$

$= \int u^{1/2}\,du$

$= \frac{2}{3} \cdot u^{3/2} + C$

$= \frac{2}{3} \cdot [\tan(x)]^{3/2} + C$

$\boxed{= \frac{2}{3} \tan^{3/2}(x) + C}$

check! $\frac{d}{dx}\left[\frac{2}{3}\tan^{3/2}(x) + C\right]$

$= \frac{2}{3} \cdot \frac{3}{2} \cdot \tan^{1/2}(x) \cdot \frac{d}{dx}[\tan(x)] + 0$

$= \tan^{1/2}(x) \cdot \sec^2(x)$

$\boxed{= \sqrt{\tan(x)}\,\sec^2(x)}$ ✓

Change of Variables

Ex.4 Integrate: $\int x\sqrt{2x+1}\,dx$

No derivative relationship.

$= \int \left(\frac{u-1}{2}\right)\cdot(u)^{1/2}\cdot\left(\frac{du}{2}\right)$

$= \frac{1}{4}\cdot\int (u-1)\cdot u^{1/2}\,du$

$= \frac{1}{4}\cdot\int (u^{3/2}-u^{1/2})\,du$

$= \frac{1}{4}\cdot\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]+C$

$= \frac{1}{10}u^{5/2}-\frac{1}{6}u^{3/2}+C$

$\boxed{= \frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{3/2}+C}$

$= \frac{6}{60}u^{5/2}-\frac{10}{60}u^{3/2}+C$

$= \frac{2}{60}u^{3/2}\cdot\left[3u^{2/2}-5\right]+C$

OR $= \frac{1}{30}u^{3/2}(3u-5)+C$

or

$= \frac{1}{30}(2x+1)^{3/2}\cdot[3(2x+1)-5]+C$

$= \frac{1}{30}(2x+1)^{3/2}\cdot(6x+3-5)+C$

$= \frac{1}{30}(2x+1)^{3/2}(6x-2)+C$

$\boxed{= \frac{1}{15}(2x+1)^{3/2}(3x-1)+C}$

"Inside"

Let $u=2x+1$, $\quad u-1=2x$

$\frac{du}{dx}=2$ $\quad \frac{u-1}{2}=x$ ★

$du=\frac{du}{dx}\cdot dx$

$du=2\,dx$

$\frac{du}{2}=dx$

check:

$\frac{d}{dx}\left[\frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{3/2}+C\right]$

$=\frac{1}{10}\cdot\frac{5}{2}\cdot(2x+1)^{3/2}\cdot 2-\frac{1}{6}\cdot\frac{3}{2}\cdot(2x+1)^{1/2}\cdot 2+0$

$=\frac{1}{2}(2x+1)^{3/2}-\frac{1}{2}(2x+1)^{1/2}$

$=\frac{1}{2}(2x+1)^{1/2}\cdot[(2x+1)^{2/2}-1]$

$=\frac{1}{2}\cdot\sqrt{2x+1}\cdot[(2x+1)-1]$

$=\frac{1}{2}\cdot\sqrt{2x+1}\cdot(2x)$

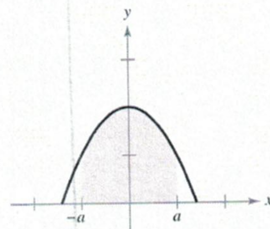
$= x\sqrt{2x+1}$ ✓

Even & Odd Functions

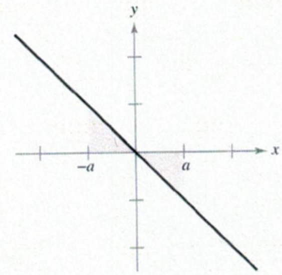
THEOREM 4.15 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
2. If f is an *odd* function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.



Even function



Odd function

Do you see a derivative?

Ex.5 Integrate: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{1+\cos(x)}}\,dx$

Let $u = 1+\cos(x)$

$\frac{du}{dx} = -\sin(x)$

$du = \frac{du}{dx}\cdot dx$

$du = -\sin(x)\,dx$

$\frac{du}{-\sin(x)} = dx$

$x = \frac{\pi}{2}$

$u = 1+\cos(\pi/2)$

$u = 1+0$

$u = 1$

$x = -\pi/2$

$u = 1+\cos(-\frac{\pi}{2})$

$u = 1+0$

$u = 1$

$$= \int_{u=1}^{u=1} \frac{\sin(x)}{\sqrt{u}} \cdot \left(\frac{du}{-\sin(x)}\right)$$

$$= -\int_{1}^{1} u^{-1/2}\,du$$

$$= -\left[\frac{2}{1} u^{1/2}\right]_1^1$$

$$= -[2-2]$$

$$\boxed{= 0} \checkmark$$

Net Area

$y = \frac{\sin(x)}{\sqrt{1+\cos(x)}} = f(x)$

↑ODD Function

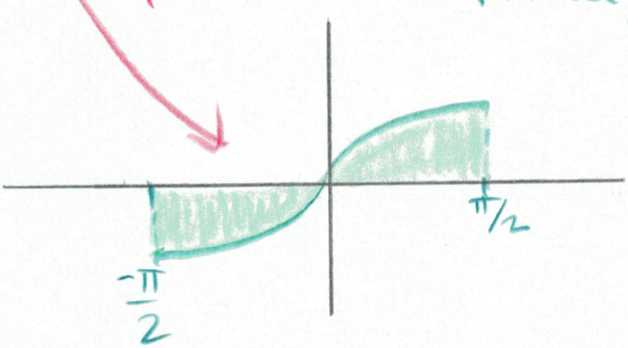
$f(-x) = -f(x)$

Check:

$f(-x) = \frac{\sin(-x)}{\sqrt{1+\cos(-x)}}$

$f(-x) = \frac{-\sin(x)}{\sqrt{1+\cos(x)}}$

$f(-x) = -f(x)$ ✓



Do you see a derivative?

Ex.6 Integrate: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2(x)\cos(x)\,dx$

$= 2\cdot\int_0^{\frac{\pi}{4}} \sin^2(x)\cos(x)\,dx$, ← since $f(x)=\sin^2(x)\cos(x)$ is an even function!

Let $u=\sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \frac{du}{dx}\cdot dx$

$du = \cos(x)\,dx$

$\frac{du}{\cos(x)} = dx$

$x=\frac{\pi}{4}$	$x=0$
$u=\sin\left(\frac{\pi}{4}\right)$	$u=\sin(0)$
$u=\frac{\sqrt{2}}{2}$	$u=0$

$= 2\cdot\int_{u=0}^{u=\frac{\sqrt{2}}{2}} (u)^2\cdot\cos(x)\cdot\left(\frac{du}{\cos(x)}\right)$

$= 2\cdot\int_0^{\frac{\sqrt{2}}{2}} u^2\,du$

$= 2\cdot\left[\frac{1}{3}u^3\right]_0^{\frac{\sqrt{2}}{2}}$

$= \frac{2}{3}\cdot\left[\left(\frac{\sqrt{2}}{2}\right)^3 - (0)^3\right]$

$= \frac{2}{3}\cdot\frac{2\sqrt{2}}{8}$

$\boxed{= \frac{\sqrt{2}}{6}}$

Net Area

check: $f(-x) = f(x)$

Even function

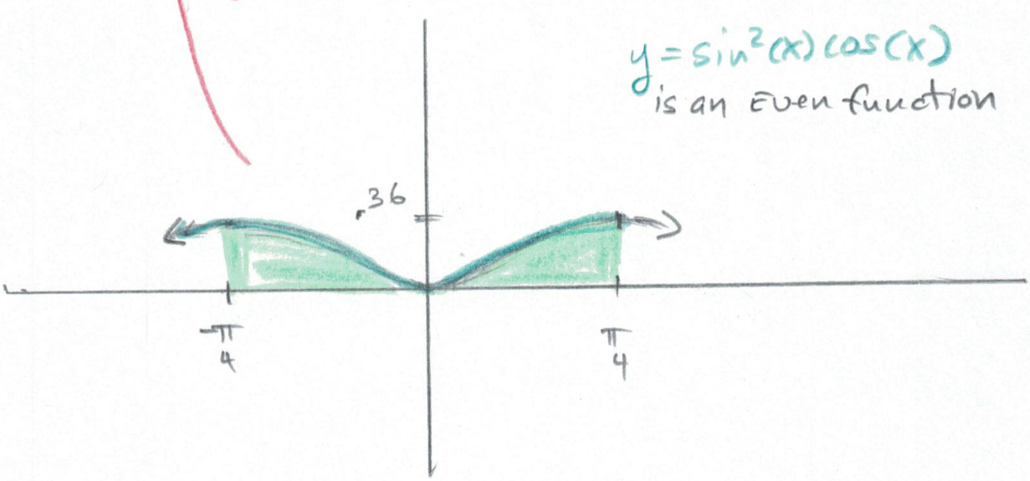
$f(-x) = \sin^2(-x)\cos(-x)$

$f(-x) = [-\sin(x)]\cdot[-\sin(x)]\cos(x)$

$f(-x) = \sin^2(x)\cos(x)$

$f(-x) = f(x)$ ✓

$y=\sin^2(x)\cos(x)$ is an Even function



Do you see a derivative?

Ex.7 Integrate: $\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)d\theta$

Let $u = \frac{1}{\theta} = \theta^{-1}$

$\frac{du}{d\theta} = -1\cdot\theta^{-2} = \frac{-1}{\theta^2}$

$du = \frac{du}{d\theta}\cdot d\theta$

$du = -\frac{1}{\theta^2}d\theta$

$-\theta^2 du = d\theta$

$= \int \frac{1}{\theta^2}\cos(u)\cdot(-\theta^2 du)$

$= -\int \cos(u)\,du$

$= -\sin(u) + C$

$\boxed{= -\sin\left(\frac{1}{\theta}\right) + C}$ ✓

Ex.8 Integrate: $\int \frac{x^2-1}{\sqrt{2x-1}}dx$ No derivative!

"Inside"

Let $u = 2x-1$

$\frac{du}{dx} = 2$

$du = \frac{du}{dx}\cdot dx$

$du = 2dx$

$\frac{du}{2} = dx$

solve for x:

$u+1 = 2x$

$\frac{u+1}{2} = x$ ★

$= \int\left(\frac{x^2-1}{\sqrt{u}}\right)\cdot\left(\frac{du}{2}\right)$

$= \frac{1}{2}\cdot\int\left[\left(\frac{u+1}{2}\right)^2 - 1\right]\cdot u^{-1/2}du$

$= \frac{1}{2}\cdot\int\left[\frac{u^2}{4} + \frac{u}{2} + \frac{1}{4} - \frac{4}{4}\right]\cdot u^{-1/2}du$

$= \frac{1}{2}\int\left[\frac{u^2}{4} + \frac{u}{2} - \frac{3}{4}\right]\cdot u^{-1/2}du$

$= \frac{1}{2}\int\left[\frac{u^{3/2}}{4} + \frac{u^{1/2}}{2} - \frac{3u^{-1/2}}{4}\right]du$

$= \frac{1}{2}\cdot\left[\frac{1}{4}\cdot\frac{2}{5}u^{5/2} + \frac{1}{2}\cdot\frac{2}{3}u^{3/2} - \frac{3}{4}\cdot\frac{2}{1}u^{1/2}\right] + C$

$= \frac{1}{2}\cdot\left[\frac{1}{10}u^{5/2} + \frac{1}{3}u^{3/2} - \frac{3}{2}u^{1/2}\right] + C$

$= \frac{1}{20}u^{5/2} + \frac{1}{6}u^{3/2} - \frac{3}{4}u^{1/2} + C$

$\boxed{= \frac{1}{20}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} - \frac{3}{4}(2x-1)^{1/2} + C}$ ✓

Section 5.4 *Logarithmic, Exponential, & other Transcendental Functions*

Ex.1 Find $\frac{dy}{dx}$: $y = x^2 e^{-x}$

"Product rule"

$\frac{d}{dx}[y] = \frac{d}{dx}[x^2 e^{-x}]$

$\frac{dy}{dx} = (x^2)\cdot\frac{d}{dx}[e^{-x}] + (e^{-x})\frac{d}{dx}(x^2)$

$\frac{dy}{dx} = x^2\cdot(e^{-x})\cdot\frac{d}{dx}[-x] + (e^{-x})\cdot(2x)$

$\frac{dy}{dx} = x^2 e^{-x}\cdot(-1) + 2xe^{-x}$

$\frac{dy}{dx} = xe^{-x}(-x+2)$, or $\boxed{\frac{dy}{dx} = xe^{-x}(2-x)}$ ✓

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Ex.2 Find $\frac{dy}{dx}$: $e^{xy} + x^2 - y^2 = 10$

$\frac{d}{dx}[e^{xy} + x^2 - y^2] = \frac{d}{dx}[10]$

$\frac{d}{dx}[e^{xy}] + \frac{d}{dx}[x^2] - \frac{d}{dx}[y^2] = 0$

$e^{xy}\cdot\frac{d}{dx}(xy) + 2x - 2y\cdot\frac{dy}{dx} = 0$

$e^{xy}\cdot[x\cdot\frac{d}{dx}(y) + y\cdot\frac{d}{dx}(x)] + 2x - 2y\cdot\frac{dy}{dx} = 0$

"Product Rule"

$e^{xy}[x\cdot\frac{dy}{dx} + y] + 2x - 2y\cdot\frac{dy}{dx} = 0$

$e^{xy}\cdot x\cdot\frac{dy}{dx} + e^{xy}\cdot y + 2x - 2y\cdot\frac{dy}{dx} = 0$

$e^{xy}y + 2x = 2y\cdot\frac{dy}{dx} - e^{xy}x\cdot\frac{dy}{dx}$

$e^{xy}y + 2x = [2y - e^{xy}x]\cdot\frac{dy}{dx}$ ← "factor out $\frac{dy}{dx}$"

$\boxed{\frac{e^{xy}y + 2x}{2y - e^{xy}x} = \frac{dy}{dx}}$ ✓

"e to the u"

Ex.3 Integrate: $\int \frac{e^{\frac{1}{x^2}}}{x^3}dx$

$= \int \frac{e^u}{x^3} \cdot \left(\frac{x^3\,du}{-2}\right)$

$= -\frac{1}{2} \cdot \int e^u\,du$

$= -\frac{1}{2} \cdot e^u + C$

$\boxed{= -\frac{1}{2} e^{\frac{1}{x^2}} + C}$ ✓

Let $u = \frac{1}{x^2} = x^{-2}$

$\frac{du}{dx} = -2x^{-3}$

$du = \frac{du}{dx} \cdot dx$

$du = -2x^{-3}dx$

$\frac{du}{-2x^{-3}} = dx$

$\frac{x^3\,du}{-2} = dx$

Do you see a derivative?

Ex.4 Integrate: $\int \frac{e^{2x}}{1+e^{2x}}dx$

$= \int \frac{e^{2x}}{(u)} \cdot \left(\frac{du}{2e^{2x}}\right)$

$= \frac{1}{2} \cdot \int \frac{1}{u}\,du$

$= \frac{1}{2} \cdot \ln|u| + C$

$= \frac{1}{2} \ln|1+e^{2x}| + C$

$= \frac{1}{2} \ln(1+e^{2x}) + C$

or

$\boxed{= \ln\left(\sqrt{1+e^{2x}}\right) + C}$ ✓

Let $u = 1+e^{2x}$

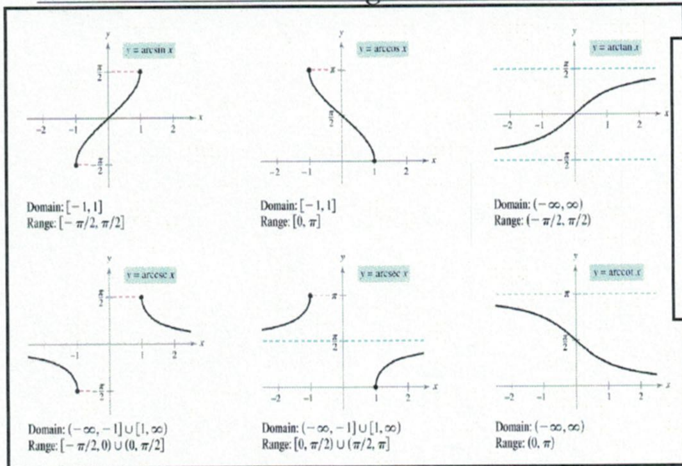
$\frac{du}{dx} = e^{2x} \cdot 2$

$du = \frac{du}{dx} \cdot dx$

$du = 2e^{2x}dx$

$\frac{du}{2e^{2x}} = dx$

Section 5.6 Inverse Trigonometric Functions: Differentiation



THEOREM 5.16 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Ex.1 Find $f'(t)$: $f(t) = \arcsin(t^2)$

$$\frac{d}{dt}[f(t)] = \frac{d}{dt}[\arcsin(t^2)]$$

$$f'(t) = \frac{1}{\sqrt{1-(t^2)^2}} \cdot \frac{d}{dt}[t^2]$$

$$f'(t) = \frac{1}{\sqrt{1-t^4}} \cdot (2t)$$

$$\boxed{f'(t) = \frac{2t}{\sqrt{1-t^4}}}$$

Let $\theta = \arcsin(t^2)$ ★★★

$\sin(\theta) = \sin[\arcsin(t^2)]$

$\sin(\theta) = t^2 = \frac{opp}{hyp} = \frac{t^2}{1}$

$$\frac{d}{dt}[\sin(\theta)] = \frac{d}{dt}[t^2]$$

$$\cos(\theta) \cdot \frac{d\theta}{dt} = 2t$$

$$\frac{d\theta}{dt} = \frac{2t}{\cos(\theta)}$$

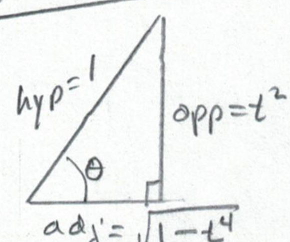
$$\cos(\theta) = \frac{adj}{hyp} = \frac{\sqrt{1-t^4}}{1}$$

$$\cos(\theta) = \sqrt{1-t^4}$$

$$\frac{d\theta}{dt} = \frac{2t}{\sqrt{1-t^4}}$$

$$\frac{d}{dt}[\arcsin(t^2)] = \frac{2t}{\sqrt{1-t^4}}$$

$$\boxed{f'(t) = \frac{2t}{\sqrt{1-t^4}}}$$



$hyp^2 = opp^2 + adj^2$

$1 = (t^2)^2 + adj^2$

$1 = t^4 + adj^2$

$1 - t^4 = adj^2$

$\sqrt{1-t^4} = adj$

★★★ STANDARD Technique for getting "rid" of the inverse notation, You will see this frequently.

Ex.2 Find $h'(x)$: $h(x) = x^2 \arctan(x)$

$$\frac{d}{dx}[h(x)] = \frac{d}{dx}[x^2 \arctan(x)]$$

$$h'(x) = [\arctan(x)] \cdot \frac{d}{dx}(x^2) + (x^2) \cdot \frac{d}{dx}[\arctan(x)]$$

$$h'(x) = [\arctan(x)] \cdot (2x) + (x^2) \cdot \left(\frac{1}{1+x^2}\right)$$

$$\boxed{h'(x) = 2x\arctan(x) + \frac{x^2}{1+x^2}}$$

or

$$\boxed{h'(x) = 2x\tan^{-1}(x) + \frac{x^2}{1+x^2}}$$

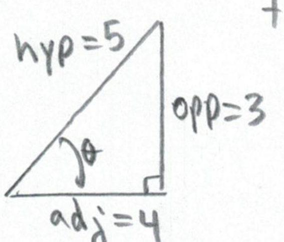
Ex.3 Evaluate: $\sin\left(\arctan\left(\frac{3}{4}\right)\right)$

Let $\theta = \arctan(\frac{3}{4})$ ★★★

$\tan(\theta) = \tan[\arctan(\frac{3}{4})]$

$\tan(\theta) = \frac{3}{4}$

$\tan(\theta) = \frac{opp}{adj} = \frac{3}{4}$



$hyp^2 = opp^2 + adj^2$

$hyp^2 = (3)^2 + (4)^2$

$hyp^2 = 9 + 16$

$hyp^2 = 25$

$hyp = 5.$

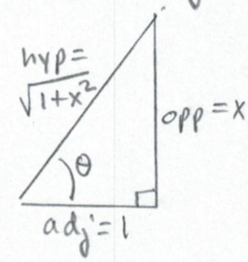
$$\sin[\arctan(\tfrac{3}{4})] = \sin(\theta) = \frac{opp}{hyp} = \frac{3}{5} \checkmark$$

Let $\theta = \arctan(x)$ ★★★

$\tan(\theta) = \tan[\arctan(x)]$

$\tan(\theta) = x$

$\tan(\theta) = \frac{opp}{adj} = \frac{x}{1}$



$hyp^2 = opp^2 + adj^2$

$hyp^2 = (x)^2 + (1)^2$

$hyp^2 = x^2 + 1$

$hyp = \sqrt{x^2+1}$

$\frac{d}{dx}[\tan(\theta)] = \frac{d}{dx}[x]$

$\sec^2(\theta) \cdot \frac{d\theta}{dx} = 1$

$\frac{d\theta}{dx} = \frac{1}{\sec^2(\theta)}$

$\sec(\theta) = \frac{hyp}{adj} = \frac{\sqrt{x^2+1}}{1}$

$\sec(\theta) = \sqrt{x^2+1}$

$\frac{d\theta}{dx} = \frac{1}{(\sqrt{x^2+1})^2}$

$\frac{d\theta}{dx} = \frac{1}{x^2+1}$

$\frac{d}{dx}[\arctan(x)] = \frac{1}{x^2+1}$

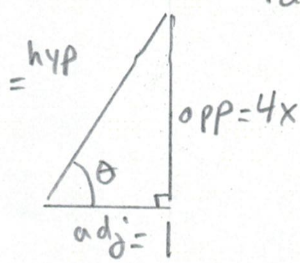
Ex.4 Write an algebraic form for $\sec(\arctan(4x))$

Let $\theta = \arctan(4x)$

$\tan(\theta) = \tan[\arctan(4x)]$

$\tan(\theta) = 4x$

$\tan(\theta) = \frac{opp}{adj'} = \frac{4x}{1}$



$hyp^2 = opp^2 + adj'^2$

$hyp^2 = (4x)^2 + (1)^2$

$hyp^2 = 16x^2 + 1$

$hyp = \sqrt{16x^2+1}$

$\sec[\arctan(4x)]$

$= \sec(\theta)$

$= \frac{hyp}{adj}$

$= \frac{\sqrt{16x^2+1}}{1}$

$\boxed{= \sqrt{16x^2+1}}$

Section 5.7 *Inverse Trigonometric Functions: Integration*

THEOREM 5.17 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

"Difference of squares"

"Sum of squares"

Ex.1 Integrate: $\int \frac{t}{t^4+16}\,dt$

sum of squares

Let $u = t^2$

$\frac{du}{dt} = 2t$

$du = \frac{du}{dt} \cdot dt$

$du = 2t\,dt$

$\frac{du}{2t} = dt$

$= \int \frac{t}{(t^2)^2 + (4)^2}\,dt$

$= \int \frac{t}{(u)^2 + (4)^2} \cdot \left(\frac{du}{2t}\right)$

$= \frac{1}{2} \cdot \int \frac{1}{u^2 + 4^2}\,du$

$= \frac{1}{2} \cdot \left[\frac{1}{a}\arctan\left(\frac{u}{a}\right)\right] + C \quad , a = 4$

$= \frac{1}{2} \cdot \left[\frac{1}{4}\arctan\left(\frac{u}{4}\right)\right] + C$

$$\boxed{= \frac{1}{8}\arctan\left(\frac{t^2}{4}\right) + C}$$

check: $\frac{d}{dt}\left[\frac{1}{8}\arctan\left(\frac{t^2}{4}\right) + C\right]$

$= \frac{1}{8} \cdot \frac{d}{dt}\left[\arctan\left(\frac{t^2}{4}\right)\right] + 0$

$= \frac{1}{8} \cdot \left[\frac{1}{1 + \left(\frac{t^2}{4}\right)^2}\right] \cdot \frac{d}{dt}\left[\frac{t^2}{4}\right]$

$= \frac{1}{8} \cdot \left[\frac{1}{1 + \frac{t^4}{16}}\right] \cdot \left[\frac{1}{4} \cdot 2t\right]$

$= \left[\frac{1}{8 + \frac{8t^4}{16}}\right] \cdot \frac{t}{2}$

$= \frac{1}{16 + \frac{16t^4}{16}}$

$= \frac{1}{16 + t^4}$ ✓

Ex.2 Integrate: $\int \frac{1}{x\sqrt{x^4-4}}\,dx$ "Difference of squares"

Let $u = x^2$

$\frac{du}{dx} = 2x$

$du = \frac{du}{dx} \cdot dx$

$du = 2x \cdot dx$

$\frac{du}{2x} = dx$

$$= \int \frac{1}{x\sqrt{(x^2)^2 - (2)^2}}\,dx$$

$$= \int \frac{1}{x\sqrt{(u)^2 - (2)^2}} \cdot \left(\frac{du}{2x}\right)$$

$$= \frac{1}{2} \cdot \int \frac{1}{x^2\sqrt{u^2 - 2^2}}\,du$$

$$= \frac{1}{2} \cdot \int \frac{1}{u\sqrt{u^2 - 2^2}}\,du$$

"Inverse Secant, Not Inverse Sine."

$a = 2$

$$= \frac{1}{2} \cdot \left[\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)\right] + C$$

$$= \frac{1}{2} \cdot \left[\frac{1}{2}\operatorname{arcsec}\left(\frac{|x^2|}{2}\right)\right] + C$$

$$\boxed{= \frac{1}{4}\operatorname{arcsec}\left(\frac{x^2}{2}\right) + C}$$ ✓

check: $\frac{d}{dx}\left[\frac{1}{4}\operatorname{arcsec}\left(\frac{x^2}{2}\right) + C\right]$

$$= \frac{1}{4} \cdot \left[\frac{1}{\left|\frac{x^2}{2}\right|\sqrt{\left(\frac{x^2}{2}\right)^2 - 1}} \cdot \frac{d}{dx}\left(\frac{x^2}{2}\right)\right] + 0$$

$$= \frac{1}{4} \cdot \left[\frac{1}{\left(\frac{x^2}{2}\right)\sqrt{\frac{x^4}{4} - 1}}\right] \cdot \left[\frac{1}{2} \cdot 2x\right]$$

$$= \frac{1}{2x\sqrt{\frac{x^4}{4} - \frac{4}{4}}} = \frac{1}{2x\,\frac{\sqrt{x^4-4}}{\sqrt{4}}} = \frac{1}{\frac{2x\sqrt{x^4-4}}{2}}$$

$$= \frac{1}{x\sqrt{x^4-4}}$$ ✓

Ex.3 Integrate: $\int \frac{2}{\sqrt{-x^2+4x}}dx$

$$= 2\cdot\int \frac{1}{\sqrt{4-(x-2)^2}}dx$$

$$= 2\cdot\int \frac{1}{(2)^2-(u)^2}\cdot du$$

"Difference of squares"

$$= 2\cdot\left[\arcsin\left(\frac{u}{a}\right)\right] + C,$$

$a = 2$

$$\boxed{= 2\arcsin\left(\frac{x-2}{2}\right) + C}$$

"complete the square"

$$-x^2+4x = -(x^2-4x)$$

$$= -(x^2-4x+\boxed{4}) + \boxed{4}$$

$$\left[\tfrac{1}{2}(-4)\right]^2 = (-2)^2 = 4$$

$$= -(x^2-4x+4) + 4$$

$$= 4 - (x^2-4x+4)$$

$$= 4 - (x-2)^2$$

Let $u = x-2$

$$\frac{du}{dx} = 1$$

$$du = \frac{du}{dx}\cdot dx$$

$$du = 1\cdot dx$$

$$du = dx$$

check:

$$\frac{d}{dx}\left[2\arcsin\left(\frac{x-2}{2}\right)+C\right]$$

$$= 2\cdot\left[\frac{1}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}}\right]\cdot\frac{d}{dx}\left(\frac{x-2}{2}\right) + 0$$

$$= \left[\frac{2}{\sqrt{1-\frac{(x^2-4x+4)}{4}}}\right]\cdot\frac{1}{2}\cdot(1-0)$$

$$= \frac{1}{\sqrt{\frac{4}{4}-\frac{x^2-4x+4}{4}}} = \frac{1}{\sqrt{\frac{4-x^2+4x-4}{4}}} = \frac{1}{\frac{\sqrt{-x^2+4x}}{\sqrt{4}}}$$

$$= \frac{1}{\frac{\sqrt{-x^2+4x}}{2}} = \frac{2}{\sqrt{-x^2+4x}}$$ ✓

Summary of Differentiation Rules

General Differentiation Rules

Let f, g, and u be differentiable functions of x.

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Sum or Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Product Rule:

$$\frac{d}{dx}[fg] = fg' + gf'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

Derivatives of Algebraic Functions

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\, u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}\, u'$$

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Basic Differentiation Rules for Elementary Functions

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u\, u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$
22. $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1 + u^2}$
23. $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

5. $\int \dfrac{du}{u} = \ln|u| + C$

6. $\int e^u\,du = e^u + C$

7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$

19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \arctan \dfrac{u}{a} + C$

20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \operatorname{arcsec} \dfrac{|u|}{a} + C$

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